Int. J. Solids Structures Vol. 23, No. 6, pp. 719-731, 1987
Printed in Great Britain.

0020-7683/87 \$3.00+.00

ASYMPTOTIC EXPANSIONS FOR LARGE ELASTIC STRAIN OF A CIRCULAR PLATE

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(Received 15 January 1986; in revised form 25 July 1986)

Abstract—A WKB-type asymptotic solution is developed for the equations governing large elastic deformation of a clamped circular plate under uniform pressure. The analysis, which neglects transverse shear strains, includes large displacements, rotations, and normal strains for a plate composed of a Mooney material. With the plate interior dominated by non-linear membrane action, the results indicate the existence of two types of edge boundary layers: a membrane-type layer, which develops as strains grow large, is contained within a wider, bending-type boundary layer. Results indicate a strong interaction between bending and stretching in the edge zone.

INTRODUCTION

Asymptotic analysis facilitates the study of boundary layer phenomena in non-linear plate problems. Previous applications of the method to circular plates have employed either the von Kármán[1-3] or the Reissner equations[4, 5]. Both of these sets of equations are, however, limited to small strains and linear constitutive relations, restrictions that have been relaxed by recently developed plate and shell theories[6-14]. Using these new theories and extending the WKB-type asymptotic expansion of Taber[5], this paper examines the effects of large strain and material nonlinearity on the behavior of a clamped circular plate with a uniform surface pressure.

The basic equations used in this work combine and specialize the developments of Reissner[6, 7, 15], Simmonds[12], and Taber[13, 14] for an isotropic, incompressible plate composed of a Mooney material. This theory, which incorporates thickness change through a modified Kirchhoff hypothesis, allows large displacement, rotation, and strain, but ignores transverse shear deformation. The asymptotic analysis of these equations for large deflections yields a first approximation solution dominated by non-linear membrane action everywhere except near the plate edge, where two types of boundary layers occur: a bending-type and a narrower, membrane-type boundary layer.

The presence of a bending boundary layer in a clamped plate under pressure has been known for many years[2]. This layer can develop when strains are small. On the other hand, the membrane boundary layer forms as the strains grow large, and so, as shown later, material nonlinearity becomes an important factor. Both boundary layers narrow as the pressure increases or the plate grows thinner.

FUNDAMENTAL EQUATIONS

Recently, Taber[14] presented equations governing the large deformation behavior of a circular plate which, for small strain, reduce to those of Reissner[15]. Kinematic quantities are based on a modified Kirchhoff hypothesis, which assumes that normals to the reference surface remain straight and normal but can change in length. In addition, shear strains that arise from the radial gradient of the plate thickness are neglected. Along with the strain energy density function developed by Simmonds[12], these equations, which are summarized in the following, provide a first approximation plate theory valid for large axisymmetric strains.

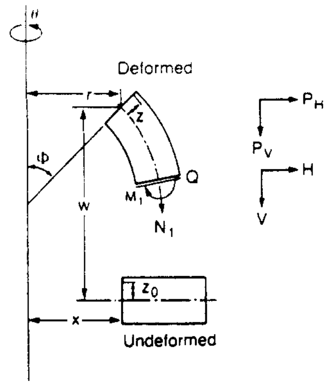

Fig. 1. Geometry and forces for undeformed and deformed plate element.

Geometry

For the geometry of Fig. 1, approximate stretch ratios are[13]

$$\Lambda_i = \lambda_i + z\kappa_i \quad (i = 1,2) \qquad \Lambda_3 = \partial z/\partial z_0 \tag{1}$$

in which z is determined by enforcing the incompressibility condition $\Lambda_1\Lambda_2\Lambda_3 = 1$ through the shell thickness, and the subscripts 1, 2, and 3 indicate the deformed Φ, θ, and z-directions, respectively. The stretch ratios at the reference surface $z = 0$ are

$$\lambda_1 = r'/\cos\Phi, \qquad \lambda_2 = r/x, \qquad \lambda_3 = 1/\lambda_1\lambda_2 \tag{2}$$

where a prime denotes differentiation with respect to the undeformed radial coordinate x, and the curvature measures are

$$\kappa_1 = \Phi', \qquad \kappa_2 = \sin\Phi/x. \tag{3}$$

Equilibrium

Membrane stress resultants N_i, transverse shear Q, and moment resultants M_i (Fig. 1) are defined per unit *undeformed* length of the reference surface, acting in the deformed coordinate directions. As in Reissner[15], it is convenient to introduce horizontal and vertical force components, H and V, such that (Fig. 1)

$$N_1 = H\cos\Phi + V\sin\Phi, \qquad Q = H\sin\Phi - V\cos\Phi. \tag{4}$$

Then, the conditions of vertical and horizontal force equilibrium and of moment equilibrium are

$$\begin{gathered}(xV)' + xp_V = 0, \qquad (xH)' - N_2 + xp_H = 0\\ (xM_1)' - \cos\Phi\, M_2 - x\lambda_1 Q = 0\end{gathered} \tag{5}$$

in which p_H and p_V are surface traction components per unit undeformed area of the reference surface. For a uniform pressure p relative to the deformed area

$$p_H = (\lambda_1\lambda_2 \sin\Phi)p, \qquad p_V = -(\lambda_1\lambda_2\cos\Phi)p \tag{6}$$

and a significant simplification can be obtained by integrating eqn (5)[1] to obtain

$$V = pr^2/2x. \tag{7}$$

Constitutive relations

A Mooney (neo-Hookean) material is characterized by the strain energy density function

$$W^* = C(I-3) \tag{8}$$

per unit undeformed volume, where C is a material constant and

$$I = \Lambda_1^2 + \Lambda_2^2 + 1/\Lambda_1^2\Lambda_2^2 \tag{9}$$

is a strain invariant. In a descent from three dimensions, a plate theory for large elastic strain requires a two-dimensional strain energy function

$$W = \int_{-t/2}^{t/2} W^* \,\mathrm{d}z_0 \tag{10}$$

where t is the undeformed plate thickness. Since this equation generally cannot be integrated in closed form, Simmonds[12] developed an expansion, which, for a material with W^* given by eqn (8), is†

$$\begin{aligned} W = Ct\{&\lambda_1^2 + \lambda_2^2 + 1/\lambda_1^2\lambda_2^2 - 3 \\ &+ (t^2/12\lambda_1^6\lambda_2^6)[(3+\lambda_1^4\lambda_2^2)\lambda_2^2\kappa_1^2 + 4\lambda_1\lambda_2\kappa_1\kappa_2 + (3+\lambda_1^2\lambda_2^4)\lambda_1^2\kappa_2^2]\}. \end{aligned} \tag{11}$$

This approximation is accurate when $(t\kappa_i)^2 \ll 1$ and contains an error of the order of the square of the transverse shear strain, which is neglected here. Based on eqns (1) and (11), the constitutive relations are

$$N_i = \partial W/\partial\lambda_i, \qquad M_i = \partial W/\partial\kappa_i \qquad (i = 1,2). \tag{12}$$

As discussed by Libai and Simmonds[8], a combination of material nonlinearity, thickness change, and a shift in the location of the reference surface relative to the geometric midsurface produces a strong coupling between bending and stretching. For a Mooney material, bending induces a net compressive stress resultant[8, 13], while stretching and compressing the reference surface decreases and increases the bending moment, respectively

† In deriving eqn (11), Simmonds[12] located the reference surface through a "dynamic consistency condition" $\int z\,\mathrm{d}z_0 = 0$, which implies that the material particles of the undeformed and deformed reference surfaces differ. If this condition is not enforced, i.e. the reference surface always consists of the same particles, then W takes a somewhat altered form. However, calculations show that, for moderately large bending and stretching, this difference in definition is not significant.

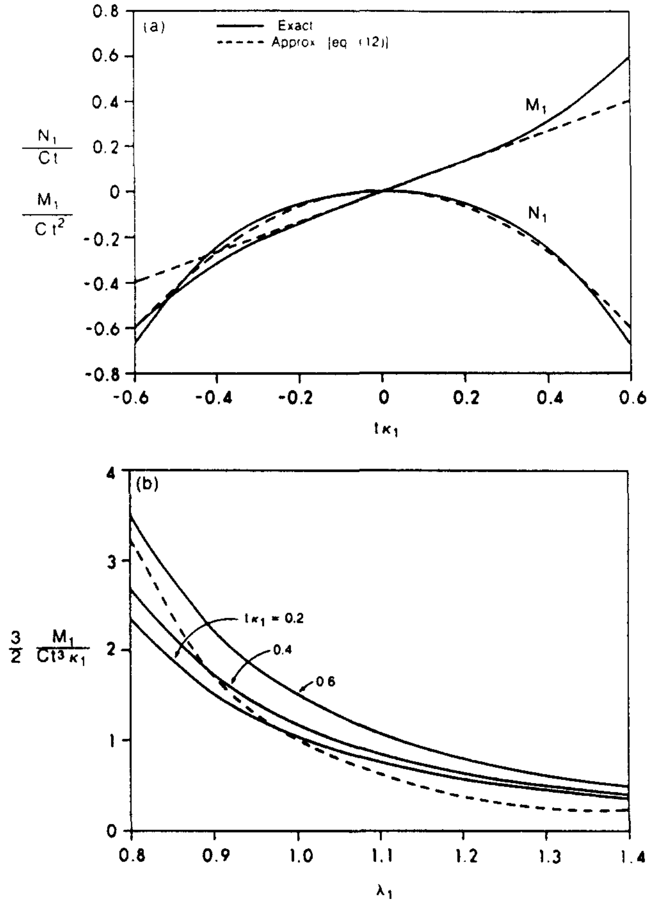

Fig. 2. Variation of stress and moment resultants at clamped edge of Mooney plate due to (a) bending and (b) stretching. "Exact" solution is based on Taber[13].

(Fig. 2). Comparison with an "exact" solution based on the modified Kirchhoff hypothesis[13] shows that, for moderately large bending ($t\kappa_i \lesssim 0.3$), these behavioral characteristics are well represented by the constitutive relations given by eqns (11) and (12). On the other hand, this form for W gives a linear moment-curvature relation (Fig. 2(a)) and does not produce the actual nonlinearities for very large bending. This limitation must be kept in mind when interpreting the results.

Non-dimensional equations

To transform the basic equations into a form suitable for asymptotic analysis, we introduce the non-dimensional quantities

$$\bar{x} = x/R, \qquad \bar{r} = r/R, \qquad \bar{\kappa}_i = R\kappa_i, \qquad \bar{M}_i = M_i/Ct^2$$
$$\bar{N}_i = N_i/pR, \qquad \bar{Q} = Q/pR, \qquad \bar{H} = H/pR, \qquad \bar{V} = V/pR \tag{13}$$

in which R is the radius of the plate and p is the uniform surface pressure. After substitution into eqns (2)–(7), (11), and (12), removal of the bars yields the non-dimensional set of equations

$$\lambda_1 = r'/\eta, \qquad \lambda_2 = r/x, \qquad \kappa_1 = \phi'/\eta, \qquad \kappa_2 = \phi/x \tag{14a}$$

$$V = r^2/2x, \qquad (xH)' - N_2 + x\lambda_1\lambda_2\phi = 0$$
$$\beta^{-1}[(xM_1)' - \eta M_2] - x\lambda_1 Q = 0 \tag{14b}$$

$$N_i = (2\lambda_i/\alpha)(1 - \lambda_i^{-4}\lambda_j^{-2}) - (\alpha/6\beta^2)\lambda_i^{-3}\lambda_j^{-2}[(1 + 9\lambda_i^{-4}\lambda_j^{-2})\kappa_i^2$$
$$+ 10(\lambda_i\lambda_j)^{-3}\kappa_i\kappa_j + (1 + 6\lambda_i^{-2}\lambda_j^{-4})\kappa_j^2]$$
$$M_i = (\alpha/6\beta)\lambda_i^{-6}\lambda_j^{-5}[(3 + \lambda_i^4\lambda_j^2)\lambda_j\kappa_i + 2\lambda_i\kappa_j], \qquad (i,j = 1,2; i \neq j) \tag{14c}$$

where

$$\phi = \sin\Phi, \qquad \eta = \cos\Phi = (1-\phi^2)^{1/2} \tag{15}$$

and

$$\alpha = pR/Ct, \qquad \beta = (R/t)\alpha. \tag{16}$$

For a clamped circular plate under uniform pressure, the linear solution is accurate up to $\beta^3/\alpha^2 = pR^4/Ct^4 \cong 10$[16], and the von Kármán equations are valid for $\alpha^2 \ll 1$[5]. The present investigation focuses on the realm of moderately large strain (up to approximately 20%) where $\alpha = O(1)$ and, for a thin plate, $R/t \gg 1$, making β a parameter large compared to unity. In addition, the relation between η and ϕ in eqn (15) limits the application to problems where $\Phi \leqslant \pi/2$ everywhere in the deformed plate. After substitution of the second of eqns (15) along with

$$N_1 = H\eta + V\phi, \qquad Q = H\phi - V\eta \tag{17}$$

as given by eqns (4), eqns (14) provide 11 relations for 11 unknowns: r, ϕ, λ_1, λ_2, κ_1, κ_2, H, V, N_2, M_1, M_2. Once those equations are solved, the vertical deflection is given by

$$w/R = \int_1^x \lambda_1\phi\,dx. \tag{18}$$

ASYMPTOTIC EXPANSIONS

For small strain, Reissner[15] reduced eqns (14) to a pair of coupled, non-linear, second-order differential equations for Φ and H. For large strain, these equations can be contracted somewhat, but at the expense of rapidly increasing complexity. Thus, to facilitate application to plates composed of materials with other types of strain energy function, eqns (14) are used as they stand. A similar procedure was followed by Wu[17], who studied boundary layer phenomena of the non-linear shell equations for small strain, and by Wu and Perng[18], who identified a boundary layer in membranes undergoing very large strains ($\alpha \gg 1$).

The dependent variables are taken in the forms of the series

$$\{a(x), \kappa_i(x)\} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \beta^{-m} e^{n\beta\xi(x)}\{a^{(mn)}(x), \beta\kappa_i^{(mn)}(x)\} \tag{19}$$

where $a = r$, ϕ, λ_i, N_i, Q, H, V, M_i, η. In these WKB-type expansions, the $n = 0$ terms provide the interior solution and the exponential ($n > 0$) terms give the edge zone solution. The $a^{(mn)}$ and $\kappa_i^{(mn)}$ are assumed to be functions of $O(1)$, and $\xi(x)$ is a decay function with $\xi(1) = 0$ and $\xi(x) < 0$ for $x < 1$. Thus, for $\beta \gg 1$, the exponential terms decay rapidly in moving away from the edge of the plate and are significant only within a narrow edge zone. This paper considers a primary (bending) boundary layer given by the $e^{\beta\xi}$ terms and a secondary (membrane) boundary layer given by the $e^{2\beta\xi}$ terms. Although higher order boundary layers are possible, these two provide the main features of the plate behavior.

After substitution of eqn (19) into eqns (14), (15), and (17), coefficients of like powers of β, $\beta e^{\beta\xi}$, etc., are equated in the usual manner. The tedious algebra was made tolerable through the use of the symbolic manipulation routine muMATH.

Interior solution

The first-order terms in β provide the system of equations

$$
\begin{aligned}
&\kappa_1^{(00)} = \kappa_2^{(00)} = M_1^{(00)} = M_2^{(00)} = 0 \\
&Q^{(00)} = H^{(00)}\phi^{(00)} - V^{(00)}\eta^{(00)} = 0 \\
&\lambda_1^{(00)} = r^{(00)\prime}/\eta^{(00)}, \qquad \lambda_2^{(00)} = r^{(00)}/x \\
&N_1^{(00)} = 2\alpha^{-1}\lambda_1^{(00)}(1 - \lambda_1^{(00)-4}\lambda_2^{(00)-2}) = H^{(00)}\eta^{(00)} + V^{(00)}\phi^{(00)} \\
&N_2^{(00)} = 2\alpha^{-1}\lambda_2^{(00)}(1 - \lambda_1^{(00)-2}\lambda_2^{(00)-4}) \\
&V^{(00)} = r^{(00)2}/2x \\
&(xH^{(00)})' - N_2^{(00)} + x\lambda_1^{(00)}\lambda_2^{(00)}\phi^{(00)} = 0 \\
&\eta^{(00)} = (1 - \phi^{(00)2})^{1/2}
\end{aligned} \tag{20}
$$

which, with vanishing bending moments and transverse shear, are the governing relations of non-linear membrane theory[19, 20]. The equations for $N_1^{(00)}$ and $Q^{(00)}$ imply the membrane conditions

$$V^{(00)} = N_1^{(00)}\phi^{(00)}, \qquad H^{(00)} = N_1^{(00)}\eta^{(00)}.$$

Equations (20) have been solved by various numerical methods since the paper of Adkins and Rivlin[19]. Here, based on the work of Tielking and Feng[21], a strain energy approach was used. With r and w taken as trigonometric series satisfying the membrane boundary conditions of vanishing edge displacement, minimization of the total potential energy provided a system of non-linear algebraic equations for the unknown coefficients which were solved by a Newton–Raphson procedure. These are the only non-linear equations that need to be solved; all other terms are given by linear systems of equations.

The equations determining the second-order terms of the interior solution are given in the Appendix. These terms, which provide only small corrections to the membrane solution when β is large, are required by the first-order edge zone solution. As discussed in the Appendix, the system of equations can be reduced to a single second-order, homogeneous, differential equation with variable coefficients for $r^{(10)}$ [eqn (A2)] which was solved by a Galerkin method.

Bending boundary layer

On substitution of eqn (19) into eqns (14), (15), and (17), the first-order $e^{\beta\xi}$ terms yield

$$
\begin{aligned}
&r^{(01)} = \lambda_1^{(01)} = \lambda_2^{(01)} = \kappa_2^{(01)} = N_1^{(01)} = N_2^{(01)} = V^{(01)} = H^{(01)} = 0 \\
&\kappa_1^{(01)} = \xi'\phi^{(01)}/\eta^{(00)} \\
&M_1^{(01)} = (\alpha/6)(3 + \lambda_1^{(00)4}\lambda_2^{(00)2})\lambda_1^{(00)-6}\lambda_2^{(00)-4}\kappa_1^{(01)} \\
&M_2^{(01)} = (\alpha/3)\lambda_1^{(00)-5}\lambda_2^{(00)-5}\kappa_1^{(01)} \\
&Q^{(01)} = H^{(00)}\phi^{(01)} - V^{(00)}\eta^{(01)} = N_1^{(00)}\phi^{(01)}/\eta^{(00)} \\
&\eta^{(01)} = -\phi^{(00)}\phi^{(01)}/\eta^{(00)}
\end{aligned} \tag{21}
$$

which contribute bending moments and transverse shear stress but not membrane stress resultants to the solution. Therefore, these terms provide the bending type of boundary layer that has been studied previously for small strains[2, 5]. After some algebraic manipulation, the moment equilibrium equation gives

$$(\xi')^2 = (6/\alpha)\lambda_1^{(00)7}\lambda_2^{(00)4}(3 + \lambda_1^{(00)4}\lambda_2^{(00)2})^{-1}N_1^{(00)} \tag{22a}$$

which corresponds to the "eikonal equation" of the WKB method. Thus, at any point in the plate, the decay function

$$\xi(x) = \int_1^x \xi'(x)\,\mathrm{d}x \tag{22b}$$

depends on the membrane solution. For small strain, $\lambda_i^{(00)} \to 1$ and in terms of dimensional quantities

$$\beta\xi(x) \cong \int_1^x (N_1^{(00)}/D)^{1/2}\,\mathrm{d}x \tag{22c}$$

where $D = 2Ct^3/3$ is the flexural rigidity for an incompressible plate. For $N_1^{(00)}$ constant, eqn (22c) corresponds to the edge zone solution for a plate strip with an applied in-plane tension, and so the asymptotic solution approximates the narrow edge zone for a circular plate with a one-dimensional boundary layer.

Once the membrane solution and $\phi^{(01)}$ are computed, all terms of the bending boundary layer can be calculated from eqns (21). But while the first-order boundary layer terms provide the additional unknown $\xi(x)$, the second-order $e^{\beta\xi}$ terms are needed for $\phi^{(01)}$. Considerable manipulation of the equations yields

$$\phi^{(01)\prime} + a\phi^{(01)} = 0 \tag{23}$$

where

$$a = \frac{1}{2}\left[\frac{1}{x} + \frac{\xi''}{\xi'} - \frac{(\eta^{(00)2})'}{\eta^{(00)2}}\right] - \frac{\lambda_1^{(00)\prime}}{\lambda_1^{(00)}} - \frac{\lambda_2^{(00)\prime}}{\lambda_2^{(00)}}$$
$$- \xi'\left(\frac{3\lambda_1^{(10)}}{2\lambda_1^{(00)}} + \frac{\lambda_2^{(10)}}{\lambda_2^{(00)}} + \frac{N_1^{(10)}}{N_1^{(00)}}\right)$$
$$- \frac{6(\lambda_1^{(00)\prime} + \xi'\lambda_1^{(10)})/\lambda_1^{(00)} + 3(\lambda_2^{(00)\prime} + \xi'\lambda_2^{(10)})/\lambda_2^{(00)}}{(3 + \lambda_1^{(00)4}\lambda_2^{(00)2})}.$$

This relation, which corresponds to the "transport equation" of the WKB solution, gives

$$\phi^{(01)} = A\exp\left(-\int_1^x a\,\mathrm{d}x\right) \tag{24}$$

where $A = \phi^{(01)}(1)$ is a constant to be determined from the boundary conditions. For a narrow boundary layer with β large, the variation of $\phi^{(01)}$ within the layer as given by eqn (24) is generally not significant, and use of the edge value is adequate. But unless stated otherwise, the full solution is examined in this paper.

Membrane boundary layer

The $e^{2\beta\xi}$ terms give

$$r^{(02)} = \lambda_2^{(02)} = V^{(02)} = H^{(02)} = \kappa_1^{(02)} = \kappa_2^{(02)} = M_1^{(02)} = M_2^{(02)} = 0$$

$$Q^{(02)} = H^{(00)}\phi^{(02)} - V^{(00)}\eta^{(02)} = 0$$

$$\lambda_1^{(02)} = \frac{\alpha^2}{24}\left(\frac{15 + \lambda_1^{(00)4}\lambda_2^{(00)2}}{3 + \lambda_1^{(00)4}\lambda_2^{(00)2}}\right)\frac{\kappa_1^{(01)2}}{\lambda_1^{(00)3}\lambda_2^{(00)2}}$$

$$\phi^{(02)} = -(\phi^{(00)}/2\eta^{(00)2})\phi^{(01)2}$$

$$N_1^{(02)} = H^{(00)}\eta^{(02)} + V^{(00)}\phi^{(02)} = N_1^{(00)}\phi^{(02)}/\phi^{(00)}$$

$$N_2^{(02)} = \frac{4}{\alpha}\frac{\lambda_1^{(02)}}{\lambda_1^{(00)3}\lambda_2^{(00)3}} - \frac{\alpha}{6}(1 + 6\lambda_1^{(00)-4}\lambda_2^{(00)-2})\frac{\kappa_1^{(01)2}}{\lambda_1^{(00)2}\lambda_2^{(00)3}}$$

$$\eta^{(02)} = -\phi^{(01)2}/2\eta^{(00)} \tag{25}$$

which contribute to the membrane stress resultants but not to the bending moments and transverse shear stress, thus giving a membrane type of boundary layer that is contained within the bending boundary layer and is half as wide. As seen in the expression for $N_2^{(02)}$, this part of the solution contains the coupling between the membrane stress resultants and the curvature change due to bending. Further discussion of these effects is presented in the results section.

First approximation solution

In summary, the non-zero terms of the asymptotic expansions provide the approximate solution

$$\begin{aligned}
r &= r^{(00)} \\
\phi &= \phi^{(00)} + e^{\beta\xi}\phi^{(01)} + e^{2\beta\xi}\phi^{(02)} \\
N_i &= N_i^{(00)} + e^{2\beta\xi}N_i^{(02)} \\
Q &= e^{\beta\xi}Q^{(01)} \\
M_i &= \beta^{-1}M_i^{(10)} + e^{\beta\xi}M_i^{(01)} \quad (i = 1,2)
\end{aligned} \tag{26}$$

in which the second-order bending moments $M_i^{(10)}$ (see Appendix) are included for the interior. These expressions clearly illustrate the primary bending-type and the secondary membrane-type boundary layers.

Boundary conditions

For a clamped plate, the regularity and boundary conditions

$$\begin{aligned}
x = 0&: \quad r = \phi = 0 \\
x = 1&: \quad r = 1, \phi = 0
\end{aligned} \tag{27}$$

must be satisfied. When the conditions at the center of the plate are considered, the edge zone solution is ignored, and eqns (26) and (27) give

$$x = 1: \quad r^{(00)} = 1, \quad \phi^{(00)} + \phi^{(01)} + \phi^{(02)} = 0. \tag{28}$$

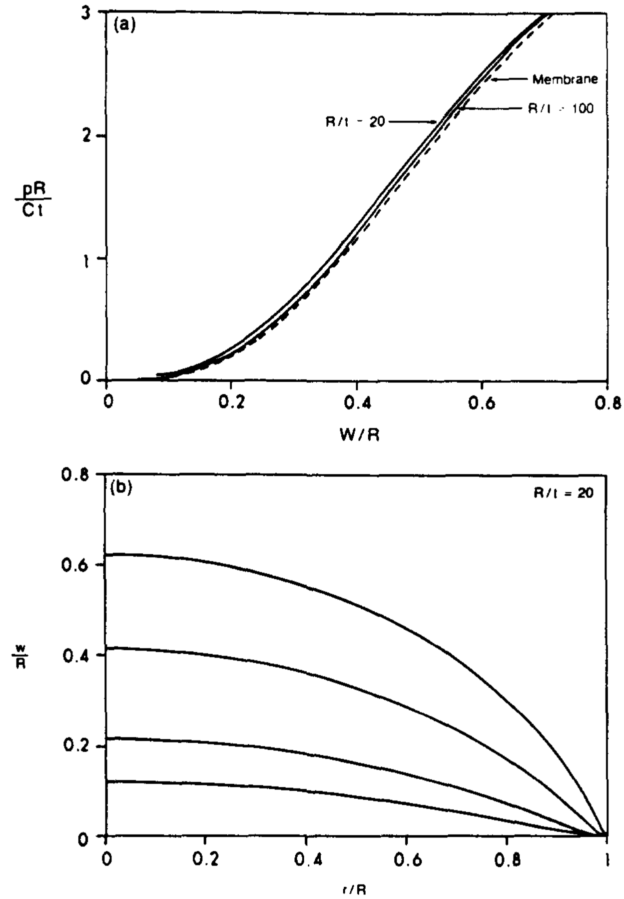

Fig. 3. (a) Load vs center deflection for clamped circular plates and a membrane. (b) Deformed profiles for clamped plate.

The first of eqns (28) is satisfied by the non-linear membrane solution, which also gives $\phi^{(00)}(1)$. Then, after substitution for $\phi^{(02)}$ from eqn (25), the second condition becomes

$$x = 1: \quad B\phi^{(01)2} + \phi^{(01)} + \phi^{(00)} = 0 \tag{29a}$$

where

$$B = -[\phi^{(00)}/2\eta^{(00)2}]_{x=1}. \tag{29b}$$

Therefore

$$\phi^{(01)}(1) = A = (1/2B)[(1 - 4B\phi^{(00)})^{1/2} - 1]_{x=1} \tag{29c}$$

provides the appropriate constant of integration in eqn (24).

RESULTS AND DISCUSSION

Load–deflection curves for clamped plates with $R/t = 20$ and 100 (Fig. 3(a)) generally show little deviation from the membrane solution. However, the curve for the thinner plate begins close to the membrane result, then approaches the curve for the thicker plate, and finally both curves move back toward the membrane solution. Note also that the load parameter $\alpha = pR/Ct$ was assumed to be $O(1)$ in the asymptotic solution; as shown in Fig. 3(a), the expansions should be valid for center deflections between approx. 20 and 60% or more of the plate radius. For this range of deflections, computed profiles for the deformed

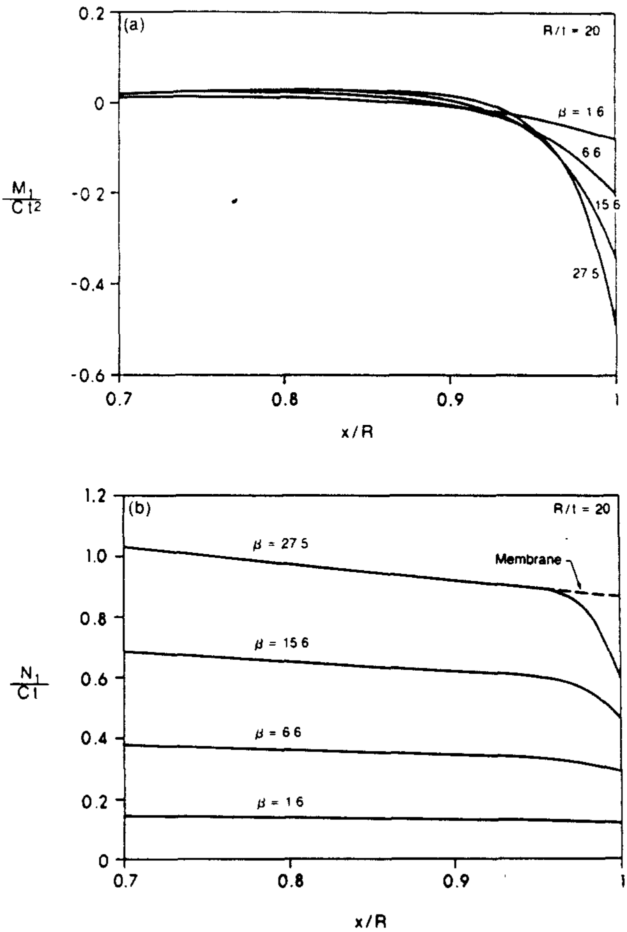

Fig. 4. Moment (a) and stress resultant (b) distributions near edge of a clamped circular plate.

plate (Fig. 3(b)) show the increasingly narrow edge zone.

Distributions of bending moment and membrane stress resultants (Fig. 4) illustrate the development of the two types of boundary layers. The non-linear membrane solution, shown for the highest load in Fig. 4(b), does not exhibit layer phenomena. But, as the load increases, the development of the primary bending and secondary membrane layers, within which the moment and stress resultants change rapidly, is clearly depicted. Outside the bending boundary layer, the second-order interior solution contributes a small moment (Fig. 4(a)).

While the bending boundary layer has received considerable attention in the past[2, 5], this type of membrane boundary layer appears to be a phenomenon unexplored until now. The fundamental plate behavior that leads to the development of this layer can be seen by examining eqns (25). First of all, the expression for $N_1^{(02)}$ derives from the deformed geometry of the edge zone and depends on $\phi^{(02)}$ and the boundary condition of zero rotation. As the deformation grows large, the horizontal component of the surface pressure becomes significant and pushes the plate outward against the support, requiring the compressive adjustment to N_1 near the edge that is provided by $N_1^{(02)}$ (Fig. 4(b)).

At the load where the membrane boundary layer first becomes noticeable, the maximum membrane and bending strains are approximately 5 and 10%, respectively. Therefore, material nonlinearity and the bending–stretching coupling illustrated in Fig. 2 come into play. Although bending alone can provide the compressive stress resultant necessary to equilibrate the boundary reaction (Fig. 2(a)), eqn (25) indicates that $\lambda_1^{(02)}$ gives an additional *stretching* of the reference surface, which counteracts some of this compressive force. This stretching adjustment comes from the constitutive relation for N_1. The bending–

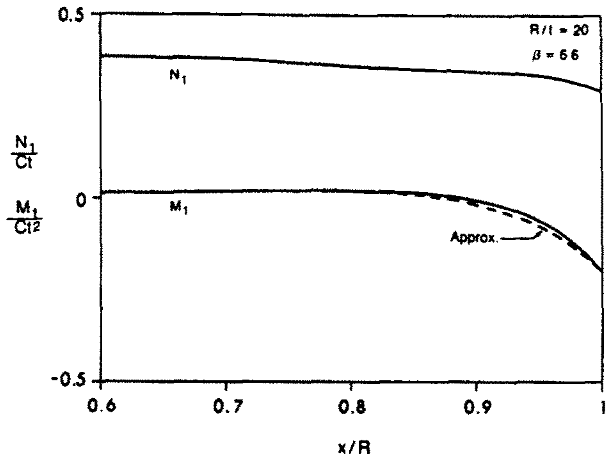

Fig. 5. Effect of constant values of $a^{(mn)}$ (approximate solution) in boundary layer terms for stress and moment resultant distributions.

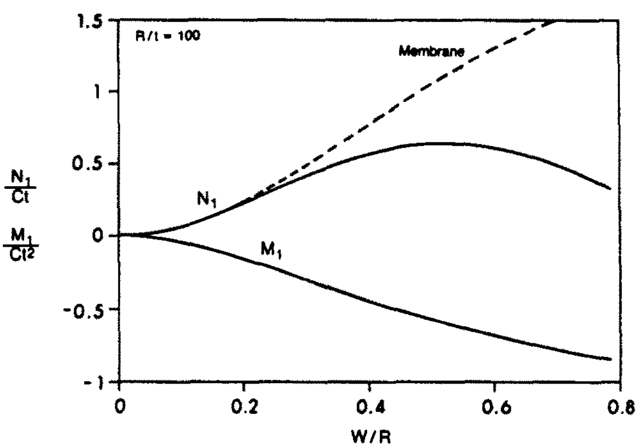

Fig. 6. Variation of stress and moment resultants with center deflection at clamped edges of circular plate and membrane.

stretching coupling, therefore, leads to complex interactions in the plate behavior within the edge zone.

As mentioned previously, the variations of the $a^{(mn)}$ terms of eqn (19) are generally not important within the edge zone if β is large. Figure 5 compares the N_1 and M_1 distributions for $\beta = 6.6$ as based on the full solution [including the transport equation (24)] and the solution using edge values for the exponential $a^{(mn)}(x)$ terms. Even for such a relatively low value for β, the N_1 curves are practically indistinguishable and the approximation for M_1 shows little error.

Finally, N_1 and M_1 at the plate edge are plotted as functions of the central plate deflection (Fig. 6). For this relatively thin plate ($R/t = 100$), N_1 initially follows close to the membrane solution. But at a deflection of about 20% of the radius, the effects of the boundary reaction to outward pressure loading become apparent, and near 50% of the radius, N_1 actually begins to decrease. The beginning of a levelling off in M_1 is due to the decrease in moment with reference surface stretching (see Fig. 2(b)). For these very large deflections, however, the bending strain is no longer small compared to unity, and the approximate expression for W [eqn (11)] is not valid. Thus, while the solution for the given equations remains valid, these results may be somewhat questionable quantitatively for the realistic situation. They do, however, substantially improve on the qualitative computations of Taber[14].

Acknowledgement—The author is grateful to Prof. James G. Simmonds for sending a copy of his manuscript [12] before its publication. This work was supported by the National Science Foundation under Grant No. MEA-8319996.

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APPENDIX

After substitution of eqns (19) into eqns (14), (15), and (17), equating coefficients of β^{-1} yields

$$
\begin{aligned}
&\lambda_1^{(10)} = (r^{(10)\prime} - \lambda_1^{(00)}\eta^{(10)})/\eta^{(00)}, \qquad \lambda_2^{(10)} = r^{(10)}/x \\
&\kappa_1^{(10)} = \phi^{(00)\prime}/\eta^{(00)}, \qquad \kappa_2^{(10)} = \phi^{(00)}/x \\
&N_i^{(10)} = (2/\alpha)[(1 + 3\lambda_i^{(00)-4}\lambda_j^{(00)-2})\lambda_i^{(10)} + 2(\lambda_i^{(00)}\lambda_j^{(00)})^{-3}\lambda_j^{(10)}] \\
&N_1^{(10)} = H^{(10)}\eta^{(00)} + V^{(10)}\phi^{(00)} \\
&M_i^{(10)} = (\alpha/6)\lambda_i^{(00)-6}\lambda_j^{(00)-5}[(3 + \lambda_i^{(00)4}\lambda_j^{(00)2})\lambda_j^{(00)}\kappa_i^{(10)} + 2\lambda_i^{(00)}\kappa_j^{(10)}] \\
&V^{(10)} = r^{(00)}r^{(10)}/x \\
&(xH^{(10)})' - N_2^{(10)} + x[\lambda_1^{(00)}\lambda_2^{(00)}\phi^{(10)} + \lambda_1^{(00)}\lambda_2^{(10)}\phi^{(00)} + \lambda_1^{(10)}\lambda_2^{(00)}\phi^{(00)}] = 0 \\
&Q^{(10)} = H^{(10)}\phi^{(00)} - V^{(10)}\eta^{(00)} + N_1^{(00)}\phi^{(10)}/\eta^{(00)} = 0 \\
&\eta^{(10)} = -\phi^{(00)}\phi^{(10)}/\eta^{(00)} \\
&\quad (i,j = 1,2; i \neq j)
\end{aligned}
\tag{A1}
$$

which, with the membrane (00) terms known, provide a linear system of equations for the second-order terms of the interior solution. This system can be reduced to a single equation of the form

$$r^{(10)\prime\prime} + C_1(x)r^{(10)\prime} + C_2(x)r^{(10)} = 0 \tag{A2}$$

subject to the boundary condition

$$x = 1: \qquad r^{(10)} + r^{(11)} + r^{(12)} = 0 \tag{A3}$$

as given by eqns (19) and (27). The asymptotic solution gives

$$r^{(11)} = -\frac{\lambda_1^{(00)}\phi^{(00)}}{\xi'\eta^{(00)}}\phi^{(01)}, \qquad r^{(12)} = \frac{1}{2\xi'}(\lambda_1^{(00)}\eta^{(02)} + \lambda_1^{(02)}\eta^{(00)}) \tag{A4}$$

which, with $\phi^{(01)}(1)$ given by eqn (29c), are known at $x = 1$ from the other pieces of the first approximation solution [see eqns (21) and (25)].